AP Calculus AB — Homework 0224 - Solutions

Dr. Paul L. Bailey — Monday, February 24, 2025

Problem 1 (1997BC.MC.NC.3)**.** Let

$$f(x) = 3x^5 - 4x^3 - 3x.$$

Find and classify the critical points of f.

Solution. If you are a Calculus student looking for critical points, you take the derivative, set it to zero, and find all real solutions. That's what you do.

$$f'(x) = 15x^4 - 12x^2 - 3 = 0 \Rightarrow 5x^4 - 4x^2 - 1 = 0 \Rightarrow (5x^2+1)(x^2-1) = 0 \Rightarrow x = 1.$$

The only (real) critical point is $x = 1$. We use the first derivative test to classify the critical point. Since $f'(0) < 0$ and $f'(2) > 0$, f changes from decreasing to increasing at $x = 1$, so f has a local minimum at $x = 1$. □

Problem 2 (1997AB.MC.NC.5)**.** Let

$$f(x) = 3x^4 - 16x^3 + 24x^2 + 48.$$

(a) Find f''.

(b) Solve $f''(x) = 0$ and create a sign chart for f''.

(c) Identity maximal intervals on which f is concave up or concave down.

Solution. We have

$$f'(x) = 12x^3 - 48x^2 + 48x \quad \text{and } f''(x) = 36x^2 - 96x + 48 = 12(3x^2 - 8x + 4) = 12(3x-2)(x-2).$$

So f'' is positive on $(-\infty, 1.5)$, negative on $(1.5, 2)$, and positive on $(2, \infty)$. So f is concave up on $(-\infty, 1.5)$, concave down on $(1.5, 2)$, and concave up on $(2, \infty)$. □

Problem 3 (Thomas §4.5 # 4)**.** A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

Solution. Let x denote the lower left corner of the base. Draw a picture and compute the area function to be

$$A : [0, \sqrt{12}] \to \mathbb{R} \quad \text{given by} \quad A(x) = 2x(12 - x^2) = 24x - 2x^3.$$

Pull a GEICO:

$$A'(x) = 24 - 6x^2 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = 2.$$

We know by construction this is a max, and $A(2) = 32$. □

Problem 4 (Thomas §4.5 # 14)**.** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000cm^3?

Solution. The volume is $V = \pi r^2 h = 1000$ and the area is $A = \pi r^2 + 2\pi r h$. We see that A is the function we wish to minimize and V is the constraint. We wish to pull another GEICO: take the derivative, set it to zero, and solve to the variable. But we have two variables! Use the constraint to eliminate one of the variable. Solve the constraint for h to get $h = \dfrac{1000}{\pi r^2}$. Plug this into the area to get $A = \pi r^2 + \dfrac{2000}{r}$. Now

$$\frac{dA}{dr} = 2\pi r - \frac{2000}{r^2} = 0 \quad \Rightarrow \quad 2\pi r^3 = 2000 \quad \Rightarrow \quad r = \frac{10}{\sqrt[3]{\pi}} \text{ and } h = r.$$

□

Problem 5 (Thomas §4.2 # 5 - 8)**.** Which functions satisfy the Mean Value Theorem on the indicated interval, and which do not? Justify you answer.

(a) $f(x) = x^{2/3}$ on $[-1, 8]$

(b) $f(x) = x^{4/5}$ on $[0, 1]$

(c) $f(x) = \sqrt{x(1-x)}$ on $[0, 1]$

(d) $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for } x \in [-\pi, 0) \\ 0 & \text{for } x = 0 \end{cases}$

Solution. **(a)** is NO because the function is not differentiable at $x = 0$. **(b)** is YES because the function IS continuous on $[0, 1]$ and IS differentiable on $(0, 1)$. **(c)** is YES for the same reason. **(d)** is NO because $\lim_{x \to 0} = f(x) = 1 \neq 0$, so the function is not continuous at $x = 0$. □

Problem 6 (Thomas §3.2 # 28)**.** Let $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$. Compute $\dfrac{dy}{dx}$.

Solution. Apply the quotient rule to get

$$\frac{dy}{dx} = \frac{12 - 6x^2}{(x^2 - 3x + 2)^2}.$$

□

Problem 7 (Thomas §5.6 # 21)**.** Compute

$$\int_0^1 \frac{12y^2 - 2y + 4}{\sqrt[3]{(4y - y^2 + 4y^3 + 1)^2}} \, dy.$$

Solution. Let $u = 4y - y^2 + 4y^3 + 1$. Then $u(0) = 1$ and $u(1) = 8$, and

$$\int_0^1 \frac{12y^2 - 2y + 4}{\sqrt[3]{(4y - y^2 + 4y^3 + 1)^2}} \, dy = \int_1^8 u^{-2/3} \, du = 3\sqrt[3]{u}\Big|_1^8 = 3(2 - 1) = 3.$$

□

Problem 8. Create an example of a function which is differentiable on $\mathbb{R}$ and whose derivative is not differentiable on $\mathbb{R}$.

Solution. Let $f(x) = x^{5/3}$. □

Problem 9. Create an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is increasing everywhere yet has infinitely many points of inflection.

Solution. Let $f(x) = \sin(x) + x$. □

Fact 1. Recall the quadratic formula: if $f(x) = ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The *discriminant* of f is

$$\Delta = b^2 - 4ac.$$

Then

- if $\Delta > 0$, f has exactly two real zeros.
- if $\Delta = 0$, then f has exactly one real zero.
- if $\Delta < 0$, then f has no real zeros.

Use this basic fact to solve the following problem.

Problem 10. Consider the cubic polynomial

$$f(x) = x^3 + ax^2 + bx.$$

Since f is a polynomial of odd degree, f has at least one real zero.

(a) Find the values of a and b for which f has exactly three zeros.

(b) Find the values of a and b for which f has exactly two zeros.

(c) Find the values of a and b for which f has exactly one zero.

(d) Find the values of a and b for which f has exactly two local extrema.

(e) Find the values of a and b for which f has exactly one horizontal tangent.

(f) Find the values of a and b for which f has no horizontal tangents.

Solution. Note that $f(x) = x(x^2 + ax + b)$. Let $\Delta = a^2 - 4b$ be the discriminate of the quadratic above. Note that $x = 0$ is a zero of f. The other two zeros are given by the quadratic formula to be

$$x = \frac{-a \pm \sqrt{\Delta}}{2}.$$

Also, $f'(x) = 3x^2 + 2ax + b$. Let $\Delta' = a^2 - 3b$ be one fourth of the discriminate of f'.

(a) $\Delta > 0$ and $a \neq \sqrt{\Delta}$

(b) $\Delta = 0$ or ($a > 0$ and $b = 0$)

(c) $\Delta < 0$ or $a^2 \leq 3b$

(d) $a^2 > 3b$

(e) $a^2 = 3b$ or $b = 0$

(f) $a^2 < 3b$

□